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Semidefinite Programming Relaxation for Nonconvex Quadratic Programs

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Abstract. This paper applies the SDP (semidefinite programming) relaxation originally developed for a 0-1 integer program to a general nonconvex QP (quadratic program) having a linear objective function and quadratic inequality constraints, and presents some fundamental characterizations of the SDP relaxation including its equivalence to a relaxation using convex-quadratic valid inequalities for the feasible region of the QP.

Key words: Semidefinite program, relaxation method, interior-point method, nonconvex quadratic program, linear matrix inequality.

1. Introduction

We use the symbols $\mathcal{S}^m$ for the set of $m \times m$ symmetric matrices, and $\mathcal{S}^m_+$ (or $\mathcal{S}^m_{++}$, respectively) for the cone consisting of $m \times m$ symmetric positive semidefinite (or positive definite, respectively) matrices. We are concerned with a canonical form QP (quadratic program):

$$\text{Minimize } \ \mathbf{c}^T\mathbf{y} \ \text{ subject to } \ \mathbf{y} \in F. \tag{1}$$

Here

$$\left.\begin{array}{rcl} F & \equiv & \left\{ \mathbf{y} \in H \ : \ \mathbf{y}^T\mathbf{P}_k\mathbf{y} \le 0 \ (k = 1, 2, \ldots, m) \right\}, \\ H & \equiv & \{ \mathbf{y} = (y_0, y_1, \ldots, y_n)^T \in R^{1+n} \ : \ y_0 = 1 \}, \\ \mathbf{c} & \equiv & \begin{pmatrix} \gamma \\ \mathbf{d} \end{pmatrix} \in R^{1+n}, \\ \mathbf{P}_k & \equiv & \begin{pmatrix} \pi_k & \mathbf{q}_k^T/2 \\ \mathbf{q}_k/2 & \mathbf{Q}_k \end{pmatrix} \in \mathcal{S}^{1+n} \ (k = 1, 2, \ldots, m), \\ \pi_k & \in & R, \ \mathbf{q}_k \in R^n, \ \mathbf{Q}_k \in \mathcal{S}^n \ (k = 1, 2, \ldots, m). \end{array}\right\} \tag{2}$$

Note that the quadratic function $\mathbf{y}^T\mathbf{P}_k\mathbf{y}$ involved in the inequality constraint is convex (strictly convex or linear, respectively) on the hyperplane H if and only if $\mathbf{Q}_k \in \mathcal{S}^n_+$ ($\mathbf{Q}_k \in \mathcal{S}^n_{++}$ or $\mathbf{Q}_k = \mathbf{O}$, respectively). $\mathbf{Q}_k$ can be indefinite, so that the feasible region F of the QP (1) is a nonconvex subset of the hyperplane H in general. This paper presents a general method for constructing an SDP (semidefinite programming) which serves as a relaxation of the QP (1). Our SDP relaxation method may be regarded as a straightforward application of the Lovász–Schrijver SDP relaxation method [16] for 0-1 IPs (integer programs) to the QP (1). In the last few years, interior-point methods originally proposed for linear programs ([11, 13, 18], etc.) were extended to SDPs ([1, 3, 4, 9, 14, 19, 29, etc.]). The extension of interior-point methods to SDPs has greatly contributed to the recent remarkable development of the SDP relaxation for various combinatorial optimization problems and nonconvex QPs ([1, 5, 6, 7, 8, 10, 12, 20, 21, 22, 23, etc.]). Among others, Ramana [21] studied the SDP relaxation for a general nonconvex QP having a quadratic objective function and quadratic inequality constraints, and Poljak, Rendl and Wolkowicz [20] studied several relaxation methods, including the SDP relaxation, for a 0-1 QP and their relations. These two works [20, 21] are closely related to the current paper; the QP (1) is a special case of the general nonconvex QP in [21], and some of the results on the 0-1 QP in [20] remain valid for the QP (1).

The canonical form QP (1) covers many mathematical programs to which the SDP relaxation has been applied so far. We can convert the 0-1 constraint $y_j = 0$ or 1 on a variable y_j into a system of quadratic inequalities $y_j(y_j - y_0) \leq 0$ and $-y_j(y_j - y_0) \leq 0$ with $y_0 = 1$. Also we can reduce the 0-1 QP studied in [20] and the general nonconvex QP in [21] into the QP (1). The important feature of the QP (1) lies in the linear objective function $\mathbf{c}^T\mathbf{y}$; specifically $\inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in \mathrm{co}F\} = \inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in F\}$, where co$F$ denotes the convex hull of F. This feature makes it possible for us to concentrate on the SDP relaxation of the feasible region F, a convex set $\widehat{F}$ including co$F$. Our major concern is how closely the SDP relaxation $\widehat{F}$ approximates co$F$.

For every $\mathbf{A} \in \mathcal{S}^m$ and $\mathbf{B} \in \mathcal{S}^m$, $\mathbf{A} \bullet \mathbf{B}$ denotes their inner product, i.e., $\mathbf{A} \bullet \mathbf{B} \equiv \mathrm{Tr}\, \mathbf{A}^T\mathbf{B}$ (the trace of $\mathbf{A}^T\mathbf{B}$). Define

$$\left.\begin{aligned} \mathbf{C} &\equiv \begin{pmatrix} \gamma & \mathbf{d}^T/2 \\ \mathbf{d}/2 & \mathbf{O} \end{pmatrix} \in \mathcal{S}^{1+n}, \\ \widehat{\mathcal{G}} &\equiv \left\{ \mathbf{Y} \in \mathcal{S}^{1+n}_+ \ : \ Y_{00} = 1,\ \mathbf{P}_k \bullet \mathbf{Y} \leq 0 \ (k = 1, 2, \ldots, m) \right\}. \end{aligned}\right\} \tag{3}$$

We now introduce an SDP:

$$\text{Minimize } \ \mathbf{C} \bullet \mathbf{Y} \ \text{ subject to } \ \mathbf{Y} \in \widehat{\mathcal{G}}. \tag{4}$$

Letting $\widehat{F} \equiv \{\mathbf{Y}\mathbf{e}_0 \ : \ \mathbf{Y} \in \widehat{\mathcal{G}}\}$, where $\mathbf{e}_0 = (1, 0, 0, \ldots, 0)^T \in R^{1+n}$, we project the SDP onto the the Euclidean space to obtain a convex minimization problem:

$$\text{Minimize } \ \mathbf{c}^T\mathbf{y} \ \text{ subject to } \ \mathbf{y} \in \widehat{F}. \tag{5}$$

Obviously, $\widehat{\mathcal{G}}$ and $\widehat{F}$ are both convex subsets of $\mathcal{S}^{1+n}$ and R^{1+n}, respectively. The lemma below shows that the two problems (4) and (5) above are equivalent, and that both serve as a relaxation of the QP (1).

LEMMA 1.1.

(i) $\mathbf{y}$ *is a feasible (or minimum, respectively) solution of problem (5) if and only if* $\mathbf{y} = \mathbf{Y}\mathbf{e}_0$ *for some feasible (or minimum, respectively) solution* $\mathbf{Y}$ *of problem (4).*

(ii) $coF \subseteq \widehat{F}$.

(iii) $\inf\{\mathbf{C} \bullet \mathbf{Y} \ : \ \mathbf{Y} \in \widehat{\mathcal{G}}\} = \inf\{\mathbf{c}^T\mathbf{y} \ : \ \mathbf{y} \in \widehat{F}\} \leq \inf\{\mathbf{c}^T\mathbf{y} \ : \ \mathbf{y} \in coF\} = \inf\{\mathbf{c}^T\mathbf{y} \ : \ \mathbf{y} \in F\}$.

Proof. The assertions (i) is straightforward from the construction of $\widehat{\mathcal{G}}$ and $\widehat{F}$. To prove (ii), assume that $\mathbf{y} \in F$. If we define $\mathbf{Y} = \mathbf{y}\mathbf{y}^T$, then $\mathbf{Y} \in \widehat{\mathcal{G}}$; hence $\mathbf{y} = \mathbf{Y}\mathbf{e}^0 \in \widehat{F}$. Since $\widehat{F}$ is convex, we obtain co $F \subseteq \widehat{F}$. The assertion (iii) follows from (i) and (ii). □

We will be mainly concerned with the convex minimization problem (5) instead of the SDP (4). If we restrict ourselves to QPs derived from 0-1 IPs, our construction of the SDP (4) and its projection (5) onto the Euclidean space are essentially the same as the ones used in the Lovász–Schrijver SDP relaxation method [16]. (This will be discussed in Section 3). They established several nice properties on their method, and applied their method and those properties to the maximum stable set problem. The assertions (i), (ii) and (iii) of Lemma 1.1 have been utilized explicitly or implicitly in many papers on the SDP relaxation. It is easily verified (and also known) that if we added the condition rank $\mathbf{Y} = 1$ in the definition of $\widehat{\mathcal{G}}$ such that

$$\widehat{\mathcal{G}} \equiv \left\{\mathbf{Y} \in \mathcal{S}^{1+n}_+ \ : \ Y_{00} = 1,\ \mathbf{P}_k \bullet \mathbf{Y} \leq 0\ (k = 1, 2, \ldots, m),\ \text{rank}\ \mathbf{Y} = 1\right\},$$

then $\widehat{F} = \{\mathbf{Y}\mathbf{e}_0 \ : \ \mathbf{Y} \in \widehat{\mathcal{G}}\}$ would coincide with the feasible region F of the QP (1). See the papers [1, 5–8, 10, 12, 20, 21, etc.].

Some characterizations of the SDP relaxation are known. If we apply Shor's relaxation [27, 28] to the QP (1), we obtain an SDP

$$\text{Maximize } t_0 \text{ subject to } \mathbf{t} \in T^d, \tag{6}$$

where

$$T^d \equiv \left\{ \mathbf{t} = (t_0, t_1, \ldots, t_m)^T \in R^{1+m} \ : \ \begin{array}{l} \mathbf{C} - t_0\mathbf{e}_0\mathbf{e}_0^T + \sum_{i=1}^{m} t_i\mathbf{P}_i \in \mathcal{S}^{1+n}_+, \\ t_i \geq 0\ (i = 1, 2, \ldots, m) \end{array} \right\}.$$

The SDP (6) is a dual of the SDP(4). To ensure the strong duality relation between the SDPs (4) and (6), we need a regularity condition (or the (generalized) Slater condition [17, 24, etc.]), Condition 1.2 below. Let

$$\begin{aligned}
\overline{\mathcal{G}} &= \{\mathbf{Y} \in \mathcal{S}^{1+n} \ : \ Y_{00} = 1,\ \mathbf{P}_k \bullet \mathbf{Y} \leq 0\ (k = 1, 2, \ldots, m)\}, \\
K &= \{k \ : \ \mathbf{P}_k \bullet \mathbf{Y} = 0 \ \text{ for every } \mathbf{Y} \in \overline{\mathcal{G}}\}, \\
K^c &= \{k \ : \ \mathbf{P}_k \bullet \mathbf{Y} < 0 \ \text{ for some } \mathbf{Y} \in \overline{\mathcal{G}}\}, \\
\mathcal{L} &= \{\mathbf{Y} \in \mathcal{S}^{1+n} \ : \ Y_{00} = 1,\ \mathbf{P}_k \bullet \mathbf{Y} = 0\ (k \in K)\}.
\end{aligned}$$

Then $\mathcal{L}$ forms the minimal affine subspace of $\mathcal{S}^{1+n}$ containing $\overline{\mathcal{G}}$.

CONDITION 1.2. The relative interior

$$\left\{ \mathbf{Y} \in \mathcal{S}^{1+n}_{++} \ : \ \begin{array}{l} Y_{00} = 1, \ \mathbf{P}_k \bullet \mathbf{Y} = 0 \ (k \in K), \\ \mathbf{P}_k \bullet \mathbf{Y} < 0 \ (k \in K^c) \end{array} \right\}$$

of the feasible region $\widehat{\mathcal{G}}$ of the SDP (4) with respect to $\mathcal{L}$ is nonempty.

By the duality theorem (see, for example, Theorem 4.2.1 of [19]) and Lemma 1.1, we obtain:

LEMMA 1.3.

(i) (Weak Duality) $\sup\{t_0 \ : \ \mathbf{t} \in T^d\} \leq \inf\{\mathbf{c}^T\mathbf{y} \ : \ \mathbf{y} \in \widehat{F}\}$.

(ii) (Strong Duality) Suppose that Condition 1.2 holds and that $-\infty < \hat{g} \equiv \inf\{\mathbf{c}^T\mathbf{y} \ : \ \mathbf{y} \in \widehat{F}\}$. *Then the SDP (6) has a maximum solution* $\mathbf{t}^* \in T^d$ *with the maximum objective value* $t_0^* = \hat{g}$.

In (ii) of Lemma 1.3, we can replace Condition 1.2 on the SDP by a stronger condition on the QP (1):

CONDITION 1.4. The interior $\{\mathbf{y} \in H \ : \ \mathbf{y}^T\mathbf{P}_k\mathbf{y} < 0 \ (k = 1, 2, \ldots, m)\}$ of the feasible region F of the QP (1) is nonempty.

In fact, we can easily verify that if $\mathbf{y}$ is an interior point of the feasible region F of the QP (1) and if $\epsilon > 0$ is sufficiently small then $(1-\epsilon)\mathbf{y}\mathbf{y}^T + \epsilon\mathbf{I}$ gives an interior point of the feasible region $\widehat{\mathcal{G}}$ of the SDP (4). Here $\mathbf{I}$ denotes the $(1+n) \times (1+n)$ identity matrix.

Poljak, Rendl and Wolkowicz [20] investigated the relation between the SDP relaxation and the Lagrange relaxation for a 0-1 QP. Some of the results presented in their paper [20] remain valid for a general nonconvex QP. We state the Lagrange relaxation for the QP (1), and relate it to the SDP relaxation for the QP (1). Define the Lagrangian function

$$L(\mathbf{y}, \mathbf{s}) = \mathbf{c}^T\mathbf{y} + \sum_{k=1}^{m} s_k \mathbf{y}^T\mathbf{P}\mathbf{y} \ \text{ for every } \mathbf{y} \in H \ \text{ and } \mathbf{s} \in R^m,$$

where $H = \{\mathbf{y} \in R^{1+n} \ : \ y_0 = 1\}$. Then, for each $\mathbf{s} \geq \mathbf{0}$, $\inf_{\mathbf{y} \in H} L(\mathbf{y}, \mathbf{s})$ gives a lower bound for the minimum value of the QP (1);

$$\inf_{\mathbf{y} \in H} L(\mathbf{y}, \mathbf{s}) \leq \inf\{\mathbf{c}^T\mathbf{y} \ : \ \mathbf{y} \in F\} \ \text{ for every } \mathbf{s} \geq \mathbf{0}.$$

It follows that

$$\sup_{\mathbf{s} \geq \mathbf{0}} \inf_{\mathbf{y} \in H} L(\mathbf{y}, \mathbf{s}) \leq \inf\{\mathbf{c}^T\mathbf{y} \ : \ \mathbf{y} \in F\}.$$

This is the standard application of the Lagrange relaxation method to the QP (1). In particular, the left hand side $\sup_{\mathbf{s}\geq\mathbf{0}} \inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s})$ of the inequality above corresponds to the Lagrangian dual of the QP (1). The Lagrangian dual $\sup_{\mathbf{s}\geq\mathbf{0}} \inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s})$ involves a hidden semidefinite constraint $\mathbf{s}\in\widetilde{T}$ on $\mathbf{s}\geq\mathbf{0}$, where

$$\widetilde{T} \equiv \{\mathbf{t}\in R^m \;:\; \mathbf{t}\geq\mathbf{0},\; \sum_{k=1}^{m} t_k\mathbf{Q}_k \in \mathcal{S}^n_+\}, \tag{7}$$

so that

$$\sup_{\mathbf{s}\geq\mathbf{0}} \inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s}) = \sup_{\mathbf{s}\in\widetilde{T}} \inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s}). \tag{8}$$

The identity above holds since $\inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s}) = -\infty$ if $\mathbf{s}\notin\widetilde{T}$ in the inner minimization of the Lagrangian dual. (This fact is due to the paper [20]). Furthermore, we can easily derive that the Lagrangian dual $\sup_{\mathbf{s}\geq\mathbf{0}} \inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s})$ is equivalent to Shor's relaxation [27, 28] of the QP (1);

$$\sup_{\mathbf{s}\in\widetilde{T}} \inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s}) = \sup\{t_0 \;:\; \mathbf{t}\in T^d\}. \tag{9}$$

Hence we obtain by Lemma 1.3 and (8) that

$$\sup_{\mathbf{s}\geq\mathbf{0}} \inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s}) \leq \inf\{\mathbf{c}^T\mathbf{y} \;:\; \mathbf{y}\in\widehat{F}\},$$

which means that the SDP relaxation of the QP (1) attains a best Lagrangian relaxation of the QP (1). This fact was shown for a 0-1 QP in the paper [20].

By Lemma 1.1, we know that the feasible region F of the QP (1) is contained in the feasible region $\widehat{F}$ of its relaxation (5). The above discussion using Shor's relaxation and the Lagrangian dual, however, has not provided us any further information on the relation between F and $\widehat{F}$. The purpose of this paper is to characterize the feasible region $\widehat{F}$ of the problem (5) in terms of *convex-quadratic valid inequalities* of the feasible region F of the QP (1).

2. Main Theorems

Let

$$\mathbf{P} \equiv \begin{pmatrix} \pi & \mathbf{q}^T/2 \\ \mathbf{q}/2 & \mathbf{Q} \end{pmatrix} \in \mathcal{S}^{1+n},\; \pi\in R,\; \mathbf{q}\in R^n,\; \mathbf{Q}\in\mathcal{S}^n.$$

We say that an inequality $\mathbf{y}^T\mathbf{P}\mathbf{y} \leq 0$ is a convex-quadratic (strictly convex-quadratic or linear, respectively) valid inequality for F if

$$\mathbf{Q}\in\mathcal{S}^n_+ \text{ } (\mathbf{Q}\in\mathcal{S}^n_{++} \text{ or } \mathbf{Q}=\mathbf{O}, \text{ respectively}) \text{ and } \mathbf{y}^T\mathbf{P}\mathbf{y}\leq 0 \text{ for every } \mathbf{y}\in F.$$

Then co$F$, the convex hull of F is completely determined by all the convex-quadratic valid inequalities (or all the linear valid inequalities) for F;

$$\mathrm{co}F = \bigcap_{\mathbf{P}\in\mathcal{V}_L} \{\mathbf{y} \in H \;:\; \mathbf{y}^T\mathbf{P}\mathbf{y} \leq 0\} = \bigcap_{\mathbf{P}\in\mathcal{V}_Q} \{\mathbf{y} \in H \;:\; \mathbf{y}^T\mathbf{P}\mathbf{y} \leq 0\},$$

where $\mathcal{V}_Q$ (or $\mathcal{V}_L$, respectively) denotes the set of all matrices $\mathbf{P} \in \mathcal{S}^{1+n}$ that induce convex-quadratic (or linear, respectively) valid inequalities for F. In general, however, it is not possible to generate all the the convex-quadratic valid inequalities (or all the linear valid inequalities) for F to form co$F$. One easy way of generating a convex-quadratic valid inequality for F is to take a nonnegative combination of the quadratic inequalities of the QP (1). This leads us to a relaxation of the QP (1) using such convex-quadratic valid inequalities for F:

$$\text{Minimize } \mathbf{c}^T\mathbf{y} \;\text{ subject to } \mathbf{y} \in \widetilde{F}, \tag{10}$$

where

$$\left.\begin{aligned}
f_{\mathbf{s}}(\mathbf{y}) &\equiv \mathbf{y}^T\left(\sum_{k=1}^{m} s_k\mathbf{P}_k\right)\mathbf{y} \;\text{ for every } \mathbf{y}\in R^{1+n} \;\text{ and every } \mathbf{s}\geq\mathbf{0},\\
\widetilde{F} &\equiv \left\{\mathbf{y}\in R^{1+n} \;:\; y_0 = 1 \;\text{ and } f_{\mathbf{s}}(\mathbf{y}) \leq 0 \;\text{ for every } \mathbf{s}\in\widetilde{T}\right\},\\
&= \left\{\mathbf{y}\in R^{1+n} \;:\; \begin{array}{l} y_0 = 1 \;\text{ and } \mathbf{y}^T\left(\displaystyle\sum_{k=1}^{m} s_k\mathbf{P}_k\right)\mathbf{y}\leq 0\\ \text{for every } \mathbf{s}\geq\mathbf{0} \;\text{ such that } \displaystyle\sum_{k=1}^{m} s_k\mathbf{Q}_k \in \mathcal{S}^n_+\end{array}\right\}.
\end{aligned}\right\} \tag{11}$$

Here $\widetilde{T}$ is defined by (7). By construction, we obviously see that $\widetilde{F}$ is a closed convex subset of H and that $F \subseteq \mathrm{co}F \subseteq \widetilde{F}$; hence the convex minimization problem (10) serves as a relaxation of the QP (1). Although the derivation of the relaxation (10) of the QP (1) is simple and straightforward, it seems difficult to implement the relaxation (10) numerically because (10) is a semi-infinite programming problem with an infinite number of convex-quadratic inequalities $f_{\mathbf{s}}(\mathbf{y}) \leq 0$ $(\mathbf{s}\in\widetilde{T})$ and the index set $\widetilde{T}$ is a continuum, non-polyhedral and convex subset of R^m in general. We may regard the right hand side $\sup_{\mathbf{s}\in\widetilde{T}} \inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s})$ of (8) as the Lagrangian dual of the semi-infinite programming problem (10). We can prove under Condition 1.4 that

$$\sup_{\mathbf{s}\in\widetilde{T}} \inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s}) = \inf\{\mathbf{c}^T\mathbf{y} \;:\; \mathbf{y}\in\widetilde{F}\}.$$

See, for example, Theorem 4.1 of [2]. Theorem 2.1 below establishes that $\widehat{F} \subseteq \widetilde{F}$ and that $\widetilde{F} = \mathrm{cl}\,\widehat{F}$, the closure of $\widehat{F}$ if Condition 1.2 holds. Thus the SDP relaxation may be regarded as an implementable version of the relaxation (10).

THEOREM 2.1.

(i) $\widehat{F} \subseteq \widetilde{F}$.

(ii) Suppose that the feasible region $\widehat{\mathcal{G}}$ *of the SDP (4) satisfies Condition 1.2. Then* $\widetilde{F} = cl\,\widehat{F}$.

Theorem 2.2 below shows that the SDP relaxation (4) (or (5)), Shor's relaxation (6), the Lagrangian dual and the relaxation (10) are equivalent under Condition 1.2.

THEOREM 2.2.

(i)
$$\begin{aligned}\sup\{t_0\ :\ \mathbf{t} \in T^d\} &= \sup_{\mathbf{0}\le\mathbf{s}\in R^m}\ \inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s}) = \sup_{\mathbf{s}\in\widetilde{T}}\ \inf_{\mathbf{y}\in H} L(\mathbf{y},\mathbf{s})\\ &\le \inf_{\mathbf{y}\in H}\ \sup_{\mathbf{s}\in\widetilde{T}} L(\mathbf{y},\mathbf{s}) = \inf\{\mathbf{c}^T\mathbf{y}\ :\ \mathbf{y}\in\widetilde{F}\}\\ &\le \inf\{\mathbf{c}^T\mathbf{y}\ :\ \mathbf{y}\in\widehat{F}\} = \inf\{\mathbf{C}\bullet\mathbf{Y}\ :\ \mathbf{Y}\in\widehat{\mathcal{G}}\}.\end{aligned}$$

(ii) If Condition 1.2 is satisfied then both inequalities "$\le$" above hold with the equality "$=$".

The proofs of Theorems 2.1 and 2.2 will be given in Section 4.

3. Some Implications of Theorem 2.1

We first consider the case where all the quadratic functions $H \ni \mathbf{y} \to \mathbf{y}^T\mathbf{P}_k\mathbf{y} \in R$ $(k = 1, 2, \dots, m)$ are convex or $\mathbf{Q}_k \in \mathcal{S}^n_+$ $(k = 1, 2, \dots, m)$. In this case, all the inequality constraints $\mathbf{y}^T\mathbf{P}_k\mathbf{y} \le 0$ $(k = 1, 2, \dots, m)$ themselves form convex-quadratic valid inequalities for F. Hence we know by Theorem 2.1 that $F = \text{co}F = \widehat{F} = \widetilde{F}$. Therefore we can compute a minimum solution of the QP (1) by solving the SDP (4).

Consider another extreme case where the quadratic functions $H \ni \mathbf{y} \to \mathbf{y}^T\mathbf{P}_k\mathbf{y} \in R$ $(k = 1, 2, \dots, \ell)$ are linear (i.e., $\mathbf{Q}_k = \mathbf{O}$ $(k = 1, 2, \dots, \ell)$) and the quadratic functions $H \ni \mathbf{y} \to \mathbf{y}^T\mathbf{P}_k\mathbf{y} \in R$ $(k = \ell+1, \ell+2, \dots, m)$ are strictly concave (i.e., $-\mathbf{Q}_k \in \mathcal{S}^n_{++}$ $(k = \ell+1, \ell+2, \dots, m)$). We then see that if $\sum_{k=1}^m s_k\mathbf{y}^T\mathbf{P}_k\mathbf{y} \le 0$ with $\mathbf{s} \ge \mathbf{0}$ is a convex-quadratic valid inequality for F then $s_k = 0$ $(k = \ell+1, \ell+2, \dots, m)$. It follows that

$$\widetilde{F} = \{\mathbf{y} \in H\ :\ \mathbf{y}^T\mathbf{P}_k\mathbf{y} \le 0\ (k = 1, 2, \dots, \ell)\}.$$

If Condition 1.2 is satisfied then cl $\widehat{F} = \widetilde{F}$. Hence the strictly concave inequality constraints $\mathbf{y}^T\mathbf{P}_k\mathbf{y} \le 0$ $(k = \ell+1, \ell+2, \dots, m)$ make no contribution to the SDP relaxation (4) in this case.

Suppose now that $\sum_{k=1}^m s_k \mathbf{y}^T \mathbf{P}_k \mathbf{y} \le 0$ with $\mathbf{s} \ge \mathbf{0}$ is a convex-quadratic valid inequality for F. By Theorem 2.1, we know that $\bar{\mathbf{y}} \notin \widehat{F}$ if $\sum_{k=1}^m s_k \bar{\mathbf{y}}^T \mathbf{P}_k \bar{\mathbf{y}} > 0$. Assume that

$$\sum_{k=1}^m s_k \bar{\mathbf{y}}^T \mathbf{P}_k \bar{\mathbf{y}} = 0 \text{ and } \bar{\mathbf{y}} \notin F. \tag{12}$$

We don't know in general whether $\bar{\mathbf{y}} \in \widehat{F}$ or $\bar{\mathbf{y}} \notin \widehat{F}$. However, if in addition $\sum_{k=1}^m s_k \mathbf{y}^T \mathbf{P}_k \mathbf{y}$ is strictly convex on H or $\sum_{k=1}^m s_k \mathbf{Q}_k \in \mathcal{S}^n_{++}$ then we can conclude that $\bar{\mathbf{y}} \notin \widehat{F}$. In fact, we see from the latter relation $\bar{\mathbf{y}} \notin F$ of (12) that $\bar{\mathbf{y}}^T \mathbf{P}_j \bar{\mathbf{y}} > 0$ for some j. If $\epsilon > 0$ is sufficiently small then $\sum_{k=1}^m s_k \mathbf{y}^T \mathbf{P}_k \mathbf{y} + \epsilon \mathbf{y}^T \mathbf{P}_j \mathbf{y} \le 0$ forms a convex-quadratic valid inequality for F which cuts off $\bar{\mathbf{y}}$, i.e., $\sum_{k=1}^m s_k \bar{\mathbf{y}}^T \mathbf{P}_k \bar{\mathbf{y}} + \epsilon \bar{\mathbf{y}}^T \mathbf{P}_j \bar{\mathbf{y}} > 0$. Hence $\bar{\mathbf{y}} \notin \widehat{F}$ by Theorem 2.1.

The discussion above suggests the following principle.

- Incorporate more "strict convexity" in the representation of the feasible region F to make the SDP relaxation (4) (or (5)) more effective.

This principle may be seen from the hidden semidefinite constraint on $\mathbf{s} \ge \mathbf{0}$ involved in the Lagrangian dual. See (8). We show some examples. Let

$$\begin{aligned} F_1 &\equiv \left\{ \mathbf{y} = (y_0, y_1, y_2)^T \in H \ : \ \begin{array}{l} -y_0(y_0 + y_j) \le 0 \ (j = 1, 2), \\ -y_0(y_0 - y_j) \le 0 \ (j = 1, 2), \\ y_0^2 - (y_1 - y_0)^2 - (y_2 - y_0)^2 \le 0 \end{array} \right\} \\ &= \left\{ \mathbf{y} = (1, y_1, y_2)^T \in R^3 \ : \ \begin{array}{l} -1 \le y_j \le 1 \ (j = 1, 2), \\ 1 - (y_1 - 1)^2 - (y_2 - 1)^2 \le 0 \end{array} \right\}, \\ H &\equiv \left\{ \mathbf{y} = (y_0, y_1, y_2)^T \in R^3 \ : \ y_0 = 1 \right\}. \end{aligned}$$

The quadratic function $y_0^2 - (y_1 - y_0)^2 - (y_2 - y_0)^2$ involved in the representation of F_1 is strictly concave on H and all other functions are linear on H. We can also verify that F_1 satisfies Condition 1.4. Thus we have that

$$\text{cl } \widehat{F}_1 = \widetilde{F}_1 = \left\{ \mathbf{y} = (1, y_1, y_2)^T \in R^3 \ : \ -1 \le y_j \le 1 \ (j = 1, 2) \right\}.$$

We can represent the same set F_1 in different ways. For example, let

$$\begin{aligned} F_2 &\equiv \left\{ \mathbf{y} = (y_0, y_1, y_2)^T \in H \ : \ \begin{array}{l} \mathbf{y}^T(\mathbf{P}_1 + \mathbf{P}_2)\mathbf{y} \le 0, \\ -y_0(y_0 + y_j) \le 0 \ (j = 1, 2), \\ -y_0(y_0 - y_j) \le 0 \ (j = 1, 2), \\ \mathbf{y}^T \mathbf{P}_3 \mathbf{y} \le 0 \end{array} \right\} \\ &= \left\{ \mathbf{y} = (1, y_1, y_2)^T \in R^3 \ : \ \begin{array}{l} -2 + y_1^2 + y_2^2 \le 0, \\ -1 \le y_j \le 1 \ (j = 1, 2), \\ 1 - (y_1 - 1)^2 - (y_2 - 1)^2 \le 0 \end{array} \right\}, \end{aligned}$$

$$F_3 \equiv \left\{ \mathbf{y} = (y_0, y_1, y_2)^T \in H \ : \ \mathbf{y}^T \mathbf{P}_k \mathbf{y} \leq 0 \ (k = 1, 2, 3) \right\}$$
$$= \left\{ \mathbf{y} = (1, y_1, y_2)^T \in R^3 \ : \ \begin{array}{l} -1 + y_j^2 \leq 0 \ (j = 1, 2), \\ 1 - (y_1 - 1)^2 - (y_2 - 1)^2 \leq 0 \end{array} \right\},$$

where

$$\mathbf{P}_1 \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{\mathbf{Q}_1} \end{pmatrix} \in \mathcal{S}^3, \ \mathbf{Q}_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}^2,$$
$$\mathbf{P}_2 \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{\mathbf{Q}_2} \end{pmatrix} \in \mathcal{S}^3, \ \mathbf{Q}_2 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{S}^2,$$
$$\mathbf{P}_3 \equiv \begin{pmatrix} -1 & 1 & 1 \\ 1 & & \\ 1 & \multicolumn{2}{c}{\mathbf{Q}_3} \end{pmatrix} \in \mathcal{S}^3, \ \mathbf{Q}_3 \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{S}^2.$$

The sets F_1, F_2 and F_3 geometrically coincide with each other although their algebraic representation are different from each other. Note also that the strictly convex-quadratic inequality constraint $\mathbf{y}^T(\mathbf{P}_1 + \mathbf{P}_2)\mathbf{y} \leq 0$ involved in the representation in F_2 is redundant. But it plays an essential role in creating a stronger SDP relaxation $\widehat{F}_2$ than $\widehat{F}_1$. In fact, we see that

$$\mathbf{y}^T (\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3) \mathbf{y} \equiv -3y_0^2 + 2y_0 y_1 + 2y_0 y_2 \leq 0 \tag{13}$$

forms a convex-quadratic valid inequality for F_2. The same convex-quadratic valid inequality (13) applies to F_3. Consequently we obtain that

$$\begin{aligned} F_1 &\subset \mathrm{co} F_1 \\ &\subset \widehat{F}_2 = \widetilde{F}_2 = \widehat{F}_3 = \widetilde{F}_3 \\ &= \left\{ \mathbf{y} = (1, y_1, y_2)^T \in R^3 \ : \ \begin{array}{l} -1 \leq y_j \leq 1 \ (j = 1, 2), \\ 2y_1 + 2y_2 \leq 3 \end{array} \right\} \\ &\subset \widehat{F}_1 = \widetilde{F}_1. \end{aligned}$$

Here "$\subset$" denotes the proper inclusion.

Now we apply Theorem 2.1 to the SDP relaxation [16] of 0-1 IPs. Consider a 0-1 IP:

$$\text{Minimize } \mathbf{c}^T \mathbf{y} \text{ subject to } \mathbf{y} \in F, \tag{14}$$

where

$$F = \left\{ \mathbf{y} = (y_0, y_1, \ldots, y_n)^T \in H \ : \ \begin{array}{l} \mathbf{a}_j^T \mathbf{y} \leq 0 \ (j = 1, 2, \ldots, \ell), \\ y_i = 0 \ \text{ or } 1 \ (i = 1, 2, \ldots, n) \end{array} \right\}.$$

We can convert this 0-1 IP into a QP:

$$\left.\begin{array}{ll}\text{Minimize} & \mathbf{c}^T\mathbf{y} \\ \text{subject to} & y_0\mathbf{a}_j^T\mathbf{y} \leq 0 \ (j = 1, 2, \ldots, \ell), \\ & y_i(y_i - y_0) \leq 0 \ (i = 1, 2, \ldots, n), \\ & -y_i(y_i - y_0) \leq 0 \ (i = 1, 2, \ldots, n) \\ & \mathbf{y} \in H. \end{array}\right\}, \tag{15}$$

which is a special case of the QP (1). We observe that if

$$\sum_{j=1}^{\ell} \lambda_j y_0 \mathbf{a}_j^T \mathbf{y} + \sum_{i=1}^{n} \mu_i y_i (y_i - y_0) + \sum_{i=1}^{n} \nu_i \left(-y_i(y_i - y_0)\right) \leq 0 \tag{16}$$

with $y_0 = 1$, $\lambda_i \geq 0$, $\mu_j \geq 0$ and $\nu_j \geq 0$ is a convex-quadratic valid inequality, we must have that $\mu_i \geq \nu_i$ $(i = 1, 2, \ldots, n)$. Hence we may assume that $\nu_i = 0$ $(i = 1, 2, \ldots, n)$ in (16). It follows that

$$\widetilde{F} = \left\{ \mathbf{y} \in H \ : \ \begin{array}{l} \mathbf{a}_j^T\mathbf{y} \leq 0 \ (j = 1, 2, \ldots, \ell), \\ 0 \leq y_i \leq 1 \ (i = 1, 2, \ldots, n) \end{array} \right\}. \tag{17}$$

Therefore the resultant relaxation problem

$$\text{Minimize } \mathbf{c}^T\mathbf{y} \ \text{ subject to } \mathbf{y} \in \widetilde{F}$$

is nothing more than the standard linear programming relaxation of the 0-1 IP (14). Note that the SDP relaxation induced from the QP (15) can satisfy Condition 1.2 although it does not satisfy Condition 1.4. In such a case, we obtain cl $\widehat{F} = \widetilde{F}$.

To make the SDP relaxation of the IP (14) effective and stronger, we need to add redundant quadratic inequality constraints such as

$$\begin{array}{l} -y_i y_k \leq 0 \ (i = 1, 2, \ldots, n, k = 1, 2, \ldots, n), \\ y_i \mathbf{a}_j^T \mathbf{y} \leq 0 \ (i = 1, 2, \ldots, n, j = 1, 2, \ldots, \ell), \\ (y_0 - y_i)\mathbf{a}_j^T\mathbf{y} \leq 0 \ (i = 1, 2, \ldots, n, j = 1, 2, \ldots, \ell), \\ -y_i(y_0 - y_k) \leq 0 \ (i = 1, 2, \ldots, n, k = 1, 2, \ldots, n), \\ -\mathbf{y}^T\mathbf{a}_j\mathbf{a}_k^T\mathbf{y} \leq 0 \ (j = 1, 2, \ldots, \ell, k = 1, 2, \ldots, \ell) \end{array}$$

to the QP (15). This was actually done in the paper [16]. See also [1] etc. Each of the additional inequalities above alone may not be a convex-quadratic valid inequality for F in general. Combining the additional inequalities together with the original ones, however, we can expect to create convex-quadratic valid inequalities for F which cut off some nonintegral vertices of the LP relaxation $\widetilde{F}$ given by (17).

4. Proof of Theorems 2.1 and 2.2

We need two lemmas to prove the theorems.

LEMMA 4.1. *Let*

$$\mathbf{P} \equiv \begin{pmatrix} \pi & \mathbf{q}^T/2 \\ \mathbf{q}/2 & \mathbf{Q} \end{pmatrix} \in \mathcal{S}^{1+n},\ \pi \in R,\ \mathbf{q} \in R^n,\ \mathbf{Q} \in \mathcal{S}^n_+.$$

Suppose that a $(1+n)\times(1+n)$ *matrix* $\mathbf{Y}$ *and a* $\mathbf{y} \in R^{1+n}$ *satisfy*

$$\mathbf{y} = \mathbf{Y}\mathbf{e}_0 \in R^{1+n},\ \mathbf{P}\bullet\mathbf{Y} \le 0,\ Y_{00} = 1 \ \textit{and} \ \mathbf{Y} \in \mathcal{S}^{1+n}_+.$$

Then $\mathbf{y} \in R^{1+n}$ *satisfies* $\mathbf{y}^T\mathbf{P}\mathbf{y} \le 0$ *and* $y_0 = 1$.

Proof. We see from $\mathbf{y} = \mathbf{Y}\mathbf{e}_0$ and $Y_{00} = 1$ that $y_0 = 1$. Let

$$\mathbf{y} = \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \text{ and } \mathbf{Y} = \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{X} \end{pmatrix}.$$

By definition and assumption, we have that

$$\begin{aligned} \mathbf{y}^T\mathbf{P}\mathbf{y} &= \pi + \mathbf{q}^T\mathbf{x} + \mathbf{x}^T\mathbf{Q}\mathbf{x} \\ &= \pi + \mathbf{q}^T\mathbf{x} + \mathbf{Q}\bullet\mathbf{X} - \mathbf{Q}\bullet(\mathbf{X} - \mathbf{x}\mathbf{x}^T) \\ &= \mathbf{P}\bullet\mathbf{Y} - \mathbf{Q}\bullet(\mathbf{X} - \mathbf{x}\mathbf{x}^T) \\ &\le -\mathbf{Q}\bullet(\mathbf{X} - \mathbf{x}\mathbf{x}^T). \end{aligned}$$

On the other hand, it follows from $\mathbf{Y} \in \mathcal{S}^{1+n}_+$ that $\mathbf{X} - \mathbf{x}\mathbf{x}^T \in \mathcal{S}^n_+$. We obtain by $\mathbf{Q} \in \mathcal{S}^n_+$ that $\mathbf{Q}\bullet(\mathbf{X} - \mathbf{x}\mathbf{x}^T) \ge 0$. Thus $\mathbf{y}^T\mathbf{P}\mathbf{y} \le 0$. □

LEMMA 4.2. $\inf\{\mathbf{c}^T\mathbf{y} \ :\ \mathbf{y} \in \widetilde{F}\} = \inf_{\mathbf{y}\in H} \sup_{\mathbf{s}\in\widetilde{T}} L(\mathbf{y},\mathbf{s})$.

Proof. Assume that $\widetilde{\mathbf{y}} \notin \widetilde{F}$. Then there is an $\widetilde{\mathbf{s}} \in \widetilde{T}$ such that $\sum_{k=1}^m \widetilde{s}_k \widetilde{\mathbf{y}}^k \mathbf{P}_k \widetilde{\mathbf{y}} > 0$. By the definition of $\widetilde{T}$, we know that $\lambda\widetilde{\mathbf{s}} \in \widetilde{T}$ for every $\lambda \ge 0$. Hence

$$\sup_{\mathbf{s}\in\widetilde{T}} L(\widetilde{\mathbf{y}},\mathbf{s}) \ \ge\ \sup_{\lambda\ge 0} \left(\mathbf{c}^T\widetilde{\mathbf{y}} + \lambda \sum_{k=1}^m \widetilde{s}_k \widetilde{\mathbf{y}}^T \mathbf{P}_k \widetilde{\mathbf{y}} \right) = +\infty.$$

Thus we have shown that $\sup_{\mathbf{s}\in\widetilde{T}} L(\mathbf{y},\mathbf{s}) = +\infty$ if $\mathbf{y} \notin \widetilde{F}$. If $\widetilde{F} = \emptyset$ then

$$\inf\{\mathbf{c}^T\mathbf{y} \ :\ \mathbf{y} \in \widetilde{F}\} = \inf_{\mathbf{y}\in H} \sup_{\mathbf{s}\in\widetilde{T}} L(\mathbf{y},\mathbf{s}) = +\infty.$$

Otherwise we have that

$$\inf_{\mathbf{y}\in H} \sup_{\mathbf{s}\in\widetilde{T}} L(\mathbf{y},\mathbf{s}) \ =\ \inf_{\mathbf{y}\in\widetilde{F}} \sup_{\mathbf{s}\in\widetilde{T}} L(\mathbf{y},\mathbf{s}) = \inf_{\mathbf{y}\in\widetilde{F}} L(\mathbf{y},\mathbf{0}) = \inf\{\mathbf{c}^T\mathbf{y} \ :\ \mathbf{y} \in \widetilde{F}\}.$$

□

Proof of Theorem 2.1. (i) Assume that $\mathbf{y} \in \widehat{F}$. Then there exists a $\mathbf{Y} \in \widehat{\mathcal{G}}$ such that $\mathbf{y} = \mathbf{Y}\mathbf{e}_0$; specifically $\mathbf{Y}$ satisfies $\mathbf{P}_k \bullet \mathbf{Y} \leq 0$ $(k = 1, 2, \ldots, m)$. Hence

$$\left(\sum_{k=1}^{m} s_k \mathbf{P}_k\right) \bullet \mathbf{Y} \leq 0 \text{ for every } \mathbf{s} = (s_1, s_2, \ldots, s_m)^T \geq \mathbf{0}.$$

By Lemma 4.1, we see that

$$\mathbf{y}^T \left(\sum_{k=1}^{m} s_k \mathbf{P}_k\right) \mathbf{y} \leq 0 \text{ whenever } \sum_{k=1}^{m} s_k \mathbf{Q}_k \in \mathcal{S}^n_+ \text{ or } \mathbf{s} \in \widetilde{T}.$$

This implies $\mathbf{y} \in \widetilde{F}$. Thus we have shown that $\widehat{F} \subseteq \widetilde{F}$.

(ii) Since $\widehat{F} \subseteq \widetilde{F}$, we know that

$$\inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in \widetilde{F}\} \leq \inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in \widehat{F}\} \tag{18}$$

for every $\mathbf{c} \in R^{1+n}$. Let $\mathbf{c} \in R^{1+n}$ be fixed arbitrarily. If $\inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in \widehat{F}\} = -\infty$ then $\inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in \widetilde{F}\} = -\infty$ by (18). Hence we obtain the equality

$$\inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in \widetilde{F}\} = \inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in \widehat{F}\}. \tag{19}$$

Now assume that $\widehat{g} \equiv \inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in \widehat{F}\} > -\infty$. By Lemma 1.3, there exists a maximum solution $\mathbf{t}^* = (t_0^*, t_1^*, \ldots, t_m^*)^T \in R^{1+m}$ of the SDP (6) with the objective value $t_0^* = \widehat{g}$. We also see by (9) and Lemma 4.2 that

$$t_0^* = \sup_{\mathbf{s} \in \widetilde{T}} \inf_{\mathbf{y} \in H} L(\mathbf{y}, \mathbf{s}) \leq \inf_{\mathbf{y} \in H} \sup_{\mathbf{s} \in \widetilde{T}} L(\mathbf{y}, \mathbf{s}) = \inf\{\mathbf{c}^T\mathbf{y} \ : \ \mathbf{y} \in \widetilde{F}\}.$$

Therefore

$$\inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in \widetilde{F}\} \ \leq \ \inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in \widehat{F}\} \ = \widehat{g} \ = t_0^* \ \leq \inf\{\mathbf{c}^T\mathbf{y} : \mathbf{y} \in \widetilde{F}\}.$$

Thus we have shown the equality (19). By the construction, $\widetilde{F}$ is a closed convex subset of R^{1+n} and $\widehat{F}$ is a convex subset of R^{1+n}. Hence the identity (19) for every $\mathbf{c} \in R^{1+n}$ implies that $\widetilde{F} = \text{cl}\ \widehat{F}$. This completes the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2. The first two equalities of the theorem follow from (8) and (9). The inequality on the second line is straightforward. The equality on the second line follows from Lemma 4.2. We obtain the inequality on the third line by $\widehat{F} \subseteq \widetilde{F}$ ((i) of Theorem 2.1), and the last equality by construction of $\widehat{F}$.

(ii) We know by Lemma 1.3 that $\sup\{t_0 \ : \ \mathbf{t} \in T^d\} = \inf\{\mathbf{c}^T\mathbf{y} \ : \ \mathbf{y} \in \widehat{F}\}$. Thus the desired result follows. $\square$

5. Concluding Remarks

If we drop the positive semidefinite constraint $\mathbf{Y} \in \mathcal{S}_+^{1+n}$ from the SDP relaxation (4), we have an LP (linear program):

$$\text{Minimize } \mathbf{C} \bullet \mathbf{Y} \ \text{ subject to } \mathbf{Y} \in \widehat{\mathcal{G}}',$$

where $\widehat{\mathcal{G}}' = \left\{ \mathbf{Y} \in \mathcal{S}^{1+n} \ : \ Y_{00} = 1 \ \text{ and } \mathbf{P}_k \bullet \mathbf{Y} \leq 0 \ (k = 1, 2, \ldots, m) \right\}$. Obviously $\widehat{\mathcal{G}} \subseteq \widehat{\mathcal{G}}'$; hence the LP serves as a relaxation of the QP (1). This LP relaxation essentially corresponds to the linearization method applied to the QP (1). See the papers [25, 26, etc.] for more details.

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